A characterization of Q-polynomial association schemes

Hirotake Kurihara, Hiroshi Nozaki

June 14, 2011

Abstract

We prove a necessary and sufficient condition for a symmetric association scheme to be a Q-polynomial scheme.

Key words: Q-polynomial association scheme, s-distance set.

1 Introduction

A *symmetric association scheme* of class d is a pair $\mathfrak{X} = (X, \{R_i\}_{i=0}^d)$, where X is a finite set and each R_i is a nonempty subset of $X \times X$ satisfying the following:

(1) $R_0 = \{(x, x) \mid x \in X\}$,

(2) $X \times X = \bigcup_{i=0}^d R_i$ and $R_i \cap R_j$ is empty if $i \neq j$,

(3) ${}^tR_i = R_i$ for any $i \in \{0, 1, \ldots, d\}$, where ${}^tR_i = \{(y, x) \mid (x, y) \in R_i\}$,

(4) for all $i, j, k \in \{0, 1, \ldots, d\}$, there exist integers p_{ij}^k such that for all $x, y \in X$ with $(x, y) \in R_k$,

$$p_{ij}^k = |\{z \in X \mid (x, z) \in R_i, (z, y) \in R_j\}|.$$

The integers p_{ij}^k are called the *intersection numbers*.

Let $\mathfrak{X}$ be a symmetric association scheme. The i-th *adjacency matrix* A_i of $\mathfrak{X}$ is the matrix with rows and columns indexed by X such that the (x, y)-entry is 1 if $(x, y) \in R_i$ or 0 otherwise. The *Bose–Mesner algebra* of $\mathfrak{X}$ is the algebra generated by the adjacency matrices $\{A_i\}_{i=0}^d$ over the complex field $\mathbb{C}$. Then $\{A_i\}_{i=0}^d$ is a natural basis of the Bose–Mesner algebra. By [2, page 59], the Bose–Mesner algebra has a second basis $\{E_i\}_{i=0}^d$ such that

(1) $E_0 = |X|^{-1}J$, where J is the all-ones matrix,

(2) $I = \sum_{i=0}^d E_i$, where I is the identity matrix,

(3) $E_iE_j = \delta_{ij}E_i$, where $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ if $i \neq j$.

2010 Mathematics Subject Classification: 05E30 (51D20).

The basis $\{E_i\}_{i=0}^d$ is called the *primitive idempotents* of $\mathfrak{X}$. We have the following equations:

$$A_i = \sum_{j=0}^{d} p_i(j) E_j, \tag{1.1}$$

$$E_i = \frac{1}{|X|} \sum_{j=0}^{d} q_i(j) A_j, \tag{1.2}$$

$$A_i A_j = \sum_{k=0}^{d} p_{ij}^k A_k, \tag{1.3}$$

$$E_i \circ E_j = \frac{1}{|X|} \sum_{k=0}^{d} q_{ij}^k E_k, \tag{1.4}$$

where $\circ$ denotes the *Hadamard product*, that is, the entry-wise matrix product. The matrices $P = (p_j(i))_{i,j=0}^d$ and $Q = (q_j(i))_{i,j=0}^d$ are called the first and second *eigenmatrices*, respectively. The numbers q_{ij}^k are called the *Krein parameters*. The Krein parameters are nonnegative real numbers (the Krein condition) [11] [2, page 69].

A symmetric association scheme is called a *P-polynomial scheme* (or a *metric scheme*) with respect to the ordering $\{A_i\}_{i=0}^d$ if for each $i \in \{0, 1, \ldots, d\}$, there exists a polynomial v_i of degree i such that $p_i(j) = v_i(p_1(j))$ for any $j \in \{0, 1, \ldots, d\}$. We say a symmetric association scheme is a P-polynomial scheme with respect to A_1 if it has the P-polynomial property with respect to some ordering $A_0, A_1, A_{i_2}, A_{i_3}, \ldots, A_{i_d}$. Dually a symmetric association scheme is called a *Q-polynomial scheme* (or a *cometric scheme*) with respect to the ordering $\{E_i\}_{i=0}^d$ if for each $i \in \{0, 1, \ldots, d\}$, there exists a polynomial v_i^* of degree i such that $q_i(j) = v_i^*(q_1(j))$ for any $j \in \{0, 1, \ldots, d\}$. Moreover a symmetric association scheme is called a Q-polynomial scheme with respect to E_1 if it has the Q-polynomial property with respect to some ordering $E_0, E_1, E_{i_2}, E_{i_3}, \ldots, E_{i_d}$. Note that both $\{v_i\}_{i=0}^d$ and $\{v_i^*\}_{i=0}^d$ form systems of orthogonal polynomials.

Throughout this paper, we use the notation $m_i = q_i(0)$ and $\theta_i^* = q_1(i)$ for $0 \le i \le d$. If an association scheme is Q-polynomial, then $\{\theta_i^*\}_{i=0}^d$ are mutually distinct because the second eigenmatrix $Q = (v_i^*(\theta_j^*))_{j,i=0}^d$ is non-singular. For a univariate polynomial f and a matrix M, we denote by $f(M^\circ)$ the matrix obtained by substituting M into f with multiplication the Hadamard product. We introduce known equivalent conditions of the Q-polynomial property of symmetric association schemes [2, page 193]. The following are equivalent:

(1) $\mathfrak{X}$ is a Q-polynomial scheme with respect to the ordering $\{E_i\}_{i=0}^d$.

(2) $(q_{1,i}^j)_{i,j=0}^d$ is an irreducible tridiagonal matrix.

(3) For each $i \in \{0, 1, \ldots, d\}$, there exists a polynomial f_i of degree i such that $E_i = f_i(E_1^\circ)$.

In the present paper, we prove a new necessary and sufficient condition for a symmetric association scheme to be Q-polynomial. Since the Q-polynomial property of a symmetric association scheme of class 1 is trivial, we assume that d is greater than 1.

Theorem 1.1. *Let $\mathfrak{X}$ be a symmetric association scheme of class $d \geq 2$. Suppose that $\{\theta_j^*\}_{j=0}^d$ are mutually distinct. Then the following are equivalent:*

(1) *$\mathfrak{X}$ is a Q-polynomial scheme with respect to E_1.*

(2) *There exists $l \in \{2, 3, \ldots, d\}$ such that for any $i \in \{1, 2, \ldots, d\}$,*

$$\prod_{\substack{j=1 \\ j \neq i}}^{d} \frac{\theta_0^* - \theta_j^*}{\theta_i^* - \theta_j^*} = -p_i(l).$$

Moreover if (2) *holds, then $l = i_d$.*

Remark 1.2. We call a finite set X in $\mathbb{R}^m$ a *d-distance set* if the number of the Euclidean distances between distinct two points in X is equal to d. Larman–Rogers–Seidel [7] proved that if the size of a two-distance set with the distances a, b $(a < b)$ is greater than $2m + 3$, then there exists a positive integer k such that $a^2/b^2 = (k-1)/k$, *i.e.* $k = b^2/(b^2 - a^2)$. Bannai–Bannai [1] proved that the ratio k of the spherical embedding of a primitive association scheme of class 2 coincides with $-p_i(2)$. The research of the present paper is motivated by [1]. For a symmetric association scheme satisfying that $\{\theta_j^*\}_{j=0}^d$ are mutually distinct, the values $K_i := \prod_{j=1, j\neq i}^{d} (\theta_0^* - \theta_j^*)(\theta_i^* - \theta_j^*)^{-1}$ $(1 \leq i \leq d)$ are the generalized Larman–Rogers–Seidel's ratios [10] of the spherical embedding of this association scheme with respect to E_1. Theorem 1.1 is an extension of Bannai–Bannai's result to Q-polynomial schemes of any class. Furthermore Theorem 1.1 is a new characterization of the Q-polynomial property on the spherical embedding of a symmetric association scheme.

At the end of this paper, we give some sufficient conditions for the integrality of K_i.

2 Proof of Theorem 1.1

First we give several lemmas that will be needed to prove Theorem 1.1.

Lemma 2.1. *For any mutually distinct real numbers $\beta_1, \beta_2, \ldots, \beta_s$, the following identity holds.*

$$\sum_{i=1}^{s} \beta_i^j \prod_{\substack{k=1 \\ k \neq i}}^{s} \frac{x - \beta_k}{\beta_i - \beta_k} = x^j$$

for any $j \in \{0, 1, \ldots, s-1\}$, where x is a variable.

Proof. For each $j \in \{0, 1, \ldots, s-1\}$, the polynomial

$$L_j(x) := \sum_{i=1}^{s} \beta_i^j \prod_{\substack{k=1 \\ k \neq i}}^{s} \frac{x - \beta_k}{\beta_i - \beta_k}$$

of degree at most $s - 1$ is known as the interpolation polynomial in the Lagrange form (see [3]). Namely, the property $L_j(\beta_i) = \beta_i^j$ holds for any $i \in \{1, 2, \ldots, s\}$. Therefore $L_j(x) = x^j$, and the lemma follows. □

We say E_j is a *component* of an element M of the Bose–Mesner algebra if $E_jM \neq 0$. Let N_h denote the set of indices j such that E_j is a component of $E_1^{\circ h}$ but not of $E_1^{\circ l}$ $(0 \leq l \leq h-1)$. Note that $N_0 = \{0\}$ and $N_1 = \{1\}$.

Lemma 2.2. *Suppose $\mathfrak{X}$ is a symmetric association scheme of class $d \geq 2$. Then the following are equivalent.*

(1) *$\mathfrak{X}$ is a Q-polynomial scheme with respect to E_1.*

(2) *The cardinality of N_d is equal to* 1.

(3) *N_d is nonempty.*

Proof. (2) $\Rightarrow$ (3): Clear.
(1) $\Rightarrow$ (2): Without loss of generality, we assume that $\mathfrak{X}$ is a Q-polynomial scheme with respect to $\{E_i\}_{i=0}^d$. By noting that $\{\theta_i^*\}_{i=0}^d$ are mutually distinct, $\{E_1^{\circ i}\}_{i=0}^d$ are linearly independent, and a basis of the Bose–Mesner algebra. We have

$$E_i = f_i(E_1^\circ) = \sum_{j=0}^{i} \alpha_{i,j} E_1^{\circ j},$$

where $\alpha_{i,j} \in \mathbb{R}$ are the coefficients of a polynomial f_i of degree i. The upper triangular matrix $(\alpha_{i,j})_{i,j=0}^d$ is non-singular because $\alpha_{i,i} \neq 0$ for each i. Since the inverse matrix $(\alpha'_{i,j})_{i,j=0}^d$ of $(\alpha_{i,j})_{i,j=0}^d$ is also an upper triangular matrix with $\alpha'_{i,i} \neq 0$ for each i, we can express

$$E_1^{\circ i} = \sum_{j=0}^{i} \alpha'_{i,j} E_j.$$

Therefore (2) follows.
(3) $\Rightarrow$ (1): First we prove that if N_i is empty for some $i \in \{1, 2, \ldots, d-1\}$, then N_{i+1} is also empty. Let $\mathcal{I} = \cup_{j=0}^{i-1} N_j$. We consider the expression $\sum_{j=0}^{i-1} E_1^{\circ j} = \sum_{j \in \mathcal{I}} \beta_j E_j$. Note that $\beta_j > 0$ for any $j \in \mathcal{I}$ by the Krein condition. Then we have

$$E_1 \circ (\sum_{h=0}^{i-1} E_1^{\circ h}) = \sum_{j \in \mathcal{I}} \beta_j \sum_{k=0}^{d} q_{1,j}^k E_k = \sum_{k=0}^{d} \sum_{j \in \mathcal{I}} \beta_j q_{1,j}^k E_k.$$

If N_i is empty, then

$$q_{1,j}^k = 0 \text{ for any } j \in \mathcal{I} \text{ and any } k \notin \mathcal{I} \tag{2.1}$$

because $\beta_j > 0$ holds for any $j \in \mathcal{I}$. We can express $E_1^{\circ i} = \sum_{j \in \mathcal{I}} \beta'_j E_j$, where β'_j are non-negative integers for any $j \in \mathcal{I}$. By (2.1) and the equalities

$$E_1^{\circ(i+1)} = E_1 \circ E_1^{\circ i} = E_1 \circ \sum_{j \in \mathcal{I}} \beta'_j E_j = \sum_{k=0}^{d} \sum_{j \in \mathcal{I}} \beta'_j q_{1,j}^k E_k,$$

we obtain $\sum_{j \in \mathcal{I}} \beta'_j q_{1,j}^k = 0$ for $k \notin \mathcal{I}$. Hence N_{i+1} is also empty. This means that if N_d is not empty, then the cardinalities of N_h is equal to 1 for any $h \in \{0, 1, \ldots, d\}$. Put $N_h = \{i_h\}$ and order $E_0, E_1, E_{i_2}, E_{i_3}, \ldots, E_{i_d}$. Then we can construct polynomials f_h of degree h such that $f_h(E_1^\circ) = E_{i_h}$ for any $h \in \{0, 1, \ldots, d\}$. Hence (1) follows. ☐

Now we prove Theorem 1.1.

Proof of Theorem 1.1. (1) $\Rightarrow$ (2): Without loss of generality, we assume that $\mathfrak{X}$ is a Q-polynomial scheme with respect to $\{E_i\}_{i=0}^d$. For each $i \in \{1, 2, \ldots, d\}$, we define the polynomial

$$F_i(t) := \prod_{\substack{j=1 \\ j \neq i}}^{d} \frac{|X|t - \theta_j^*}{\theta_i^* - \theta_j^*}$$

of degree $d-1$. Set $M_i = F_i(E_1^\circ)$. Then $|X|E_1 = \sum_{j=0}^d \theta_j^* A_j$ yields that the (x, y)-entries of M_i are

$$M_i(x,y) = \begin{cases} K_i & \text{if } (x,y) \in R_0, \\ 1 & \text{if } (x,y) \in R_i, \\ 0 & \text{otherwise,} \end{cases}$$

where $K_i := \prod_{j=1, j\neq i}^d (\theta_0^* - \theta_j^*)(\theta_i^* - \theta_j^*)^{-1}$. Since F_i is a polynomial of degree $d-1$, the matrix M_i is a linear combination of $\{E_i\}_{i=0}^{d-1}$. This means that $M_i E_d = 0$. By (1.1),

$$0 = M_i E_d = (K_i I + A_i) E_d = (K_i + p_i(d)) E_d$$

for any $i \in \{1, 2, \ldots, d\}$. Therefore the desired result follows.

(2) $\Rightarrow$ (1): From the equation $A_i = \sum_{j=0}^d p_i(j) E_j$ and our assumptions, we have

$$A_i E_l = p_i(l) E_l = -K_i E_l.$$

By Lemma 2.1,

$$(|X|E_1)^{\circ j} E_l = \big((\theta_0^*)^j I + \sum_{i=1}^d (\theta_i^*)^j A_i\big) E_l = \big((\theta_0^*)^j - \sum_{i=1}^d (\theta_i^*)^j K_i\big) E_l = 0$$

for any $j \leq d-1$. This means that l is not an element of N_j for any $j \leq d-1$. Note that the following equality holds:

$$\prod_{j=1}^d \frac{|X|E_1 - \theta_j^* J}{\theta_0^* - \theta_j^*} = I,$$

where the multiplication is the Hadamard product. Obviously, I has E_l as a component. Since $l \notin N_i$ for any $i \in \{0, 1, \ldots, d-1\}$, we have $l \in N_d$. By Lemma 2.2, the desired result follows. $\square$

3 Integrality of K_i

In this section, we consider when $K_i = -p_i(d)$ is an integer for each $i \in \{1, 2, \ldots, d\}$ for a Q-polynomial scheme. The following theorem is important in this section.

Theorem 3.1 (Suzuki [12]). *Let $\mathfrak{X}$ with $m_1 > 2$ be a Q-polynomial scheme with respect to the ordering $\{E_i\}_{i=0}^d$. Suppose $\mathfrak{X}$ is Q-polynomial with respect to another ordering. Then the new ordering is one of the following:*

(1) $E_0, E_2, E_4, E_6, \dots, E_5, E_3, E_1$,

(2) $E_0, E_d, E_1, E_{d-1}, E_2, E_{d-2}, E_3, E_{d-3}, \dots$,

(3) $E_0, E_d, E_2, E_{d-2}, E_4, E_{d-4}, \dots, E_{d-5}, E_5, E_{d-3}, E_3, E_{d-1}, E_1$,

(4) $E_0, E_{d-1}, E_2, E_{d-3}, E_4, E_{d-5}, \dots, E_5, E_{d-4}, E_3, E_{d-2}, E_1, E_d$, *or*

(5) $d = 5$ *and* $E_0, E_5, E_3, E_2, E_4, E_1$.

Note that Q-polynomial schemes with $m_1 = 2$ are the ordinary n-gons as distance-regular graphs.

Proposition 3.2. *Let $\mathfrak{X}$ with $m_1 > 2$ be a Q-polynomial association scheme with respect to the ordering $\{E_i\}_{i=0}^d$. If there exists t such that $t \leq d/2$, $t \equiv 1 \pmod 2$ and $m_t \neq m_{d-t+1}$, then K_j is an integer for any j.*

Proof. Let $\mathbb{F}$ be the splitting field of the scheme, generated by the entries of the first eigenmatrix P. Then $\mathbb{F}$ is a Galois extension of the rational field. Let G be the Galois group $\mathrm{Gal}(\mathbb{F}/\mathbb{Q})$. We consider the action of G on the primitive idempotents E_i, where elements of G are applied entry-wise. Then the action of G on $\{E_i\}_{i=0}^d$ is faithful and $|G| \leq 2$ [9].

Suppose K_j is not an integer for some j. Since $-K_j = p_j(d)$ is an eigenvalue of A_j, K_j is an algebraic integer. By the basic number theory, K_j is irrational. Therefore $|G| \neq 1$ and hence $|G| = 2$. Let σ be the non-identity element of G. From the definition of K_j, E_1 must have an irrational entry, and $E_1^\sigma \neq E_1$. Therefore $\{E_i^\sigma\}_{i=0}^d$ is another Q-polynomial ordering with the same polynomials f_i. Hence $\{E_i^\sigma\}_{i=0}^d$ coincides with one of (1)–(5) in Theorem 3.1.

For $d = 2$, it is known that K_i is an integer for each $i = 1, 2$ if $m_1 \neq m_2$ [1]. For (1) and (2) with $d > 2$, $(E_1^\sigma)^\sigma \neq E_1$, this contradicts that σ^2 is the identity. Since $p_j(d)$ is irrational and $A_j E_d = p_j(d) E_d$, E_d has an irrational entry. Therefore $E_d^\sigma \neq E_d$. For (4), σ fixes E_d, a contradiction. Therefore the ordering $\{E_i^\sigma\}_{i=0}^d$ coincides with (3) or (5).

Suppose that there exists t such that $t \leq d/2$, $t \equiv 1 \pmod 2$ and $m_t \neq m_{d-t+1}$. Since $E_t \circ I = (m_t/|X|)I$, we have $E_t^\sigma \circ I^\sigma = (m_t/|X|)I^\sigma$ and hence $E_t^\sigma \circ I = (m_t/|X|)I \neq (m_{d-t+1}/|X|)I$. Therefore $E_t^\sigma \neq E_{d-t+1}$. Thus, the ordering $\{E_i^\sigma\}_{i=0}^d$ does not coincide with (3) for $d \geq 2$. If $d = 5$, then $m_1 \neq m_5$ and hence $E_1^\sigma \neq E_5$. Therefore $\{E_i^\sigma\}_{i=0}^5$ does not coincide with (5). Thus the proposition follows. $\square$

Remark that the known Q-polynomial schemes with some irrational K_i and $d > 2$ are the ordinary n-gons and the association scheme obtained from the icosahedron [5, 8]. We can give a similar equivalent condition of the P-polynomial property of symmetric association schemes [6]. Let $\theta_i = p_1(i)$ for $0 \leq i \leq d$.

Theorem 3.3. *Let $\mathfrak{X}$ be a symmetric association scheme of class $d \geq 2$. Suppose $\{\theta_j\}_{j=0}^d$ are mutually distinct. Then the following are equivalent:*

(1) *$\mathfrak{X}$ is a P-polynomial association scheme with respect to A_1.*

(2) *There exists* $l \in \{2, 3, \ldots, d\}$ *such that for any* $i \in \{1, 2, \ldots d\}$,

$$\prod_{\substack{j=1 \\ j \neq i}}^{d} \frac{\theta_0 - \theta_j}{\theta_i - \theta_j} = -q_i(l).$$

Moreover if (2) *holds, then* $l = i_d$.

Acknowledgments. Both of authors are supported by the fellowship of the Japan Society for the Promotion of Science. The authors would like to thank Eiichi Bannai, Edwin van Dam, Tatsuro Ito, William J. Martin, Akihiro Munemasa, Hiroshi Suzuki, Makoto Tagami, Hajime Tanaka, Paul Terwilliger and Paul-Hermann Zieschang for useful discussions and comments.

References

[1] E. Bannai and E. Bannai, A note on the spherical embeddings of strongly regular graphs, *European J. Combin.* 26 (2005), no. 8, 1177–1179.

[2] E. Bannai and T. Ito, *Algebraic Combinatorics I: Association Schemes*, Benjamin/Cummings, Menro Park, CA, 1984.

[3] J.-P. Berrut and L.N. Trefethen, Barycentric Lagrange Interpolation, *SIAM Review* 46 (3) (2004), 501–517.

[4] P. Delsarte, J.M. Goethals, and J.J. Seidel, Spherical codes and designs, *Geom. Dedicata* 6 (1977), no. 3, 363–388.

[5] M. Kiyota and H. Suzuki, Character products and Q-polynomial group association schemes, *J. Algebra* 226 (2000), no. 1, 533–546.

[6] H. Kurihara and H. Nozaki, A spectral equivalent condition of the P-polynomial property for association schemes, in preparation.

[7] D.G. Larman, C.A. Rogers, and J.J. Seidel, On two-distance sets in Euclidean space, *Bull. London Math. Soc.* 9 (1977), 261–267.

[8] W.J. Martin, M. Muzychuk and J. Williford, Imprimitive cometric association schemes: constructions and analysis, *J. Algebraic Combin.* 25 (2007), no. 4, 399–415.

[9] W.J. Martin and J.S. Williford, There are finitely many Q-polynomial association schemes with given first multiplicity at least three, *European J. Combin.* 30 (3) (2009) 698–704,

[10] H. Nozaki, A generalization of Larman–Rogers–Seidel's theorem, *Discrete Math.* 311 (2011), 792–799.

[11] L.L. Scott, A conditions on Higman's parameters, *Amer. Math. Soc. Notices* 701 (1973), 20–45.

[12] H. Suzuki, Association schemes with multiple Q-polynomial structures, *J. Algebraic Combin.* 7 (2) (1998), 181–196

Hirotake Kurihara
Mathematical Institute,
Tohoku University
Aramaki-Aza-Aoba 6-3,
Aoba-ku,
Sendai 980-8578,
Japan
sa9d05@math.tohoku.ac.jp

Hiroshi Nozaki
Graduate School of Information Sciences,
Tohoku University
Aramaki-Aza-Aoba 6-3-09,
Aoba-ku,
Sendai 980-8579,
Japan
nozaki@ims.is.tohoku.ac.jp